

# CHARACTERS AND DESCENTS

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**ABSTRACT.** We study the extent to which the values attained by a symmetric group character determine the distribution of descent sets among the basis elements of the representation space. We present an explicit formula expressing the number of combinatorial objects with a given descent set in terms of the character values. Examples of basis elements include standard Young tableaux of a given shape and permutations with a fixed inversion number. Applications include a proof of the equivalence of the Foata-Schützenberger equi-distribution theorem and a theorem of Lusztig and Stanley in invariant theory. Proofs involve the analysis of a new family of asymmetric matrices of Walsh-Hadamard type.

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## 1. INTRODUCTION

Many character formulas involve the descent set of a permutation or of a standard Young tableau. We propose here a general setting for such formulas, and study the extent to which the character values determine the distribution of descent sets among the basis elements of the representation space. An explicit formula is given for the distribution of descents over certain combinatorial sets, as an alternating weighted sum of character values. Examples of such sets include signed and unsigned permutations of fixed length, involutions, standard Young tableaux, and more. It also follows that certain statements in permutation statistics have equivalent formulations in character theory. For example, the fundamental equi-distribution Theorem of Foata and Schützenberger, independently proved by Garsia and Gessel, is equivalent to a theorem of Lusztig and Stanley in invariant theory.

The key tool for obtaining these results is a certain new family of asymmetric matrices of Walsh-Hadamard type. Some of the interesting properties of these matrices are used in this paper, and some others are dealt with in [4].

The organization of this paper is as follows: Sections 2 and 3 contain the necessary definitions and background material; the main motivating question is stated at the end of Section 3. Section 4 introduces the main tool – a family (actually, two “coupled” families) of square matrices, and states key results about their determinants and (conjectured) eigenvalues. Section 5 contains a proof of the invertibility of these matrices, using Möbius inversion, as well as explicit formulas for (essentially) the entries of the inverse matrices. A general setting for the relevant character formulas is described in Section 6. Then, in Section 7, the results about the matrices are applied to give an explicit formula for the number of combinatorial objects with a given descent set in terms of the character values, and to prove the equivalence of statements in two different fields.

## 2. PRELIMINARIES AND NOTATION

### 2.1. Intervals, compositions, partitions and runs.

For positive integers  $m, n$  denote

$$[m, n] := \begin{cases} \{m, m+1, \dots, n\}, & \text{if } m \leq n; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Denote also  $[n] := [1, n] = \{1, \dots, n\}$ .

A *composition* of a positive integer  $n$  is a vector  $\mu = (\mu_1, \dots, \mu_t)$  of positive integers such that  $\mu_1 + \dots + \mu_t = n$ . A *partition* of  $n$  is a composition with weakly

decreasing entries  $\mu_1 \geq \dots \geq \mu_t > 0$ . The *underlying partition* of a composition is obtained by reordering the entries in weakly decreasing order.

For each composition  $\mu = (\mu_1, \dots, \mu_t)$  of  $n$  define the set of its partial sums

$$S(\mu) := \{\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \dots + \mu_t = n\} \subseteq [n],$$

as well as its complement

$$I(\mu) := [n] \setminus S(\mu) \subseteq [n-1].$$

For example, for the composition  $\mu = (3, 4, 2, 5)$  of 14:  $S(\mu) = \{3, 7, 9, 14\}$  and  $I(\mu) = \{1, 2, 4, 5, 6, 8, 10, 11, 12, 13\}$ .

The *runs* (maximal consecutive intervals) in  $I(\mu)$  correspond to the components of  $\mu$  satisfying  $\mu_k > 1$ ; the length of the run corresponding to  $\mu_k$  is  $\mu_k - 1$ .

## 2.2. Permutations, Young tableaux and descent sets.

Let  $S_n$  be the symmetric group on the letters  $1, \dots, n$ . For  $1 \leq i \leq n-1$  denote  $s_i := (i, i+1)$ , a simple reflection (adjacent transposition) in  $S_n$ . For a composition  $\mu = (\mu_1, \dots, \mu_t)$  of  $n$  let

$$s_\mu := (1, 2, \dots, \mu_1)(\mu_1 + 1, \mu_1 + 2, \dots, \mu_1 + \mu_2) \cdots \in S_n,$$

a product of  $t$  cycles of lengths  $\mu_1, \mu_2, \dots, \mu_t$  consisting of consecutive letters. The permutation  $s_\mu$  may be obtained from the product  $s_1 s_2 \cdots s_{n-1}$  of all simple reflections (in the usual order) by deleting the factors  $s_{\mu_1 + \dots + \mu_k}$  for all  $1 \leq k < t$ .

The *descent set* of a permutation  $\pi \in S_n$  is  $\text{Des}(\pi) := \{i : \pi(i) > \pi(i+1)\}$ .

The *descent set* of a standard Young tableaux  $T$  is the set  $\text{Des}(T) := \{1 \leq i \leq n-1 : i+1 \text{ lies southwest of } i\}$ .

## 2.3. $\mu$ -Unimodality.

A sequence  $(a_1, \dots, a_n)$  of distinct positive integers is *unimodal* if there exists  $1 \leq m \leq n$  such that  $a_1 > a_2 > \dots > a_m < a_{m+1} < \dots < a_n$ . (This definition differs slightly from the commonly used one, where all inequalities are reversed.)

Let  $\mu = (\mu_1, \dots, \mu_t)$  be a composition of  $n$ . A sequence of  $n$  positive integers is  $\mu$ -*unimodal* if the first  $\mu_1$  integers form a unimodal sequence, the next  $\mu_2$  integers form a unimodal sequence, and so on. A permutation  $\pi \in S_n$  is  $\mu$ -*unimodal* if the sequence  $(\pi(1), \dots, \pi(n))$  is  $\mu$ -unimodal. For example,  $\pi = 936871254$  is  $(4, 3, 2)$ -unimodal, but not  $(5, 4)$ -unimodal.

Let  $U_\mu$  be the set of all  $\mu$ -unimodal permutations in  $S_n$ .

## 3. A FAMILY OF CHARACTER FORMULAS

Let  $\lambda$  and  $\mu$  be partitions of  $n$ . Let  $\chi^\lambda$  be the  $S_n$ -character of the irreducible representation  $S^\lambda$ , and let  $\chi_\mu^\lambda$  be its value on a conjugacy class of cycle type  $\mu$ . The following formula for the irreducible characters is a special case of [13, Theorem 4]. For a direct combinatorial proof see [12].

**Theorem 3.1.** [13, Theorem 4]

$$\chi_\mu^\lambda = \sum_{\pi \in \mathcal{C} \cap U_\mu} (-1)^{|\text{Des}(\pi) \cap I(\mu)|},$$

where  $\mathcal{C}$  is any Knuth class of RSK-shape  $\lambda$ .

Let  $\chi^{(k)}$  be the  $S_n$ -character defined by the symmetric group action on the  $k$ -th homogeneous component of the coinvariant algebra. Then

**Theorem 3.2.** [1, Theorem 5.1]

$$\chi_\mu^{(k)} = \sum_{\pi \in L(k) \cap U_\mu} (-1)^{|\text{Des}(\pi) \cap I(\mu)|},$$

where  $L(k)$  is the set of all permutations of length  $k$  in  $S_n$ .

A complex representation of a group or an algebra  $A$  is called a *Gelfand model* for  $A$  if it is equivalent to the multiplicity free direct sum of all the irreducible  $A$ -representations. Let  $\chi^G$  be the character of the Gelfand model of  $S_n$  (or of its group algebra).

**Theorem 3.3.** [3, Theorem 1.2.3]

$$\chi_\mu^G = \sum_{\pi \in I_n \cap U_\mu} (-1)^{|\text{Des}(\pi) \cap I(\mu)|},$$

where  $I_n := \{\sigma \in S_n : \sigma^2 = \text{id}\}$  is the set of all involutions in  $S_n$ .

For more character formulas of this type see, e.g., [2, 7, 13].

In this paper we propose a general setting for all of these results. In particular, we provide an answer to the following question.

**Question 3.4.** Are these character formulas invertible? In other words, to what extent do the character values  $\{\chi_\mu^* : \forall \mu\}$  determine the distribution of descent sets among the basis elements of the representation space?

#### 4. TWO FAMILIES OF MATRICES

It turns out that the answer to the above question is deeply connected to the properties of a certain family  $(A_n)$  of square matrices. In fact, we shall define two “coupled” families of matrices,  $(A_n)$  and  $(B_n)$ . For each nonnegative integer  $n$ , both  $A_n$  and  $B_n$  are square matrices of order  $2^n$ , with entries  $0, \pm 1$ , which may be viewed as asymmetric variants of Walsh-Hadamard matrices.

We shall give two equivalent definitions for these matrices. The explicit non-recursive definition is closer in spirit to the subsequent applications, but the recursive definition is very simple to describe and easy to use, and will therefore be presented first.

#### 4.1. A recursive definition.

Recall the well known *Walsh-Hadamard* matrices, defined by the recursion

$$H_n = \begin{pmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{pmatrix} \quad (n \geq 1)$$

with  $H_0 = (1)$ .

**Definition 4.1.** Define, recursively,

$$A_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & -B_{n-1} \end{pmatrix} \quad (n \geq 1)$$

with  $A_0 = (1)$ , and

$$B_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ 0 & -B_{n-1} \end{pmatrix} \quad (n \geq 1)$$

with  $B_0 = (1)$ .

Each of the matrices  $A_n$  and  $B_n$  may be obtained from the corresponding Walsh-Hadamard matrix  $H_n$ , all the entries of which are  $\pm 1$ , by replacing some of the entries by 0.

**Example 4.2.**

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad B_1 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 \end{pmatrix} \quad B_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

#### 4.2. An explicit definition.

It will be convenient to index the rows and columns of  $A_n$  and  $B_n$  by subsets of the set  $\{1, \dots, n\}$ .

**Definition 4.3.** Let  $P_n$  be the power set (set of all subsets) of  $[n] := \{1, \dots, n\}$ . Endow  $P_n$  with the anti-lexicographic linear order: for  $I, J \in P_n$ ,  $I \neq J$ , let  $m$  be the largest element in the symmetric difference  $I \triangle J := (I \cup J) \setminus (I \cap J)$ , and define:  $I < J \iff m \in J$ .

**Example 4.4.** The linear order on  $P_3$  is

$$\emptyset < \{1\} < \{2\} < \{1, 2\} < \{3\} < \{1, 3\} < \{2, 3\} < \{1, 2, 3\}.$$

**Definition 4.5.** For  $I \in P_n$  let  $I_1, \dots, I_t$  be the sequence of *runs* (maximal consecutive intervals) in  $I$ , namely:  $I$  is the disjoint union of the  $I_k$  ( $1 \leq k \leq t$ ), and each  $I_k$  is a nonempty set of the form  $\{m_k + 1, m_k + 2, \dots, m_k + \ell_k\}$  with

$\ell_k \geq 1$  ( $\forall k$ ) and  $0 \leq m_1 < m_1 + \ell_1 < m_2 < m_2 + \ell_2 < \dots < m_t < m_t + \ell_t \leq n$ . In particular,  $|I| = \ell_1 + \dots + \ell_t$ .

**Example 4.6.** For  $I = \{1, 2, 4, 5, 6, 8, 10\} \in P_{10}$ :  $I_1 = \{1, 2\}$ ,  $I_2 = \{4, 5, 6\}$ ,  $I_3 = \{8\}$ ,  $I_4 = \{10\}$ .

Order  $P_n$  as in Definition 4.3. The entries of the Walsh-Hadamard matrix  $H_n = (h_{I,J})_{I,J \in P_n}$  are explicitly given by the formula

$$h_{I,J} := (-1)^{|I \cap J|} \quad (\forall I, J \in P_n).$$

**Definition 4.7.** A *prefix* of an interval  $I = \{m+1, \dots, m+\ell\}$  is an interval of the form  $\{m+1, \dots, m+p\}$ , for  $0 \leq p \leq \ell$ .

**Lemma 4.8.** (Explicit definition) *Order  $P_n$  as in Definition 4.3, and let  $I_1, \dots, I_t$  be the runs of  $I \in P_n$ . Then:*

(i)  $A_n = (a_{I,J})_{I,J \in P_n}$ , where

$$a_{I,J} = \begin{cases} (-1)^{|I \cap J|}, & \text{if } I_k \cap J \text{ is a prefix of } I_k \text{ for each } k; \\ 0, & \text{otherwise.} \end{cases}$$

(ii)  $B_n = (b_{I,J})_{I,J \in P_n}$ , where:

$$b_{I,J} = \begin{cases} (-1)^{|I \cap J|}, & \text{if } I_k \cap J \text{ is a prefix of } I_k \text{ for each } k, \text{ and} \\ & n \notin I \setminus J; \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* It will be convenient here to define  $A_n$  and  $B_n$  explicitly as in the lemma, and then show that they satisfy the recursions in Definition 4.1.

We shall start with  $A_n$ . Clearly  $A_0 = (1)$ .

For  $I, J \in P_n$  ( $n \geq 1$ ) denote  $I' := I \setminus \{n\}$  and  $J' := J \setminus \{n\}$ .

The “upper left” quarter of  $A_n$  corresponds to  $I, J \in P_n$  such that  $n \notin I$  and  $n \notin J$ . In this case, clearly  $a_{I,J}$  in  $A_n$  is the same as  $a_{I',J'}$  in  $A_{n-1}$ .

Similarly when  $n \notin I$  and  $n \in J$ , and also when  $n \in I$  and  $n \notin J$ :  $|I \cap J| = |I' \cap J'|$ , and  $I_k \cap J$  is a prefix of  $I_k$  for all  $k$  if and only if  $I'_k \cap J$  is a prefix of  $I'_k$  for all  $k$ .

The “lower right” quarter of  $A_n$  corresponds to  $I, J \in P_n$  such that  $n \in I \cap J$ . If  $n-1 \notin I$  then  $I_k \cap J$  is a prefix of  $I_k$  for all  $k$  if and only if  $I'_k \cap J$  is a prefix of  $I'_k$  for all  $k$ . Also  $|I \cap J| = |I' \cap J'| + 1$ , so that  $a_{I,J}$  in  $A_n$  is equal to  $-a_{I',J'}$  in  $A_{n-1}$  and also to  $-b_{I',J'}$  in  $B_{n-1}$  (since  $n-1 \notin I'$  so  $n-1 \notin I' \setminus J'$ ). If  $n-1 \in I \cap J$  then, again,  $a_{I,J}$  in  $A_n$  is equal to  $-a_{I',J'}$  in  $A_{n-1}$  and also to  $-b_{I',J'}$  in  $B_{n-1}$  (since  $n-1 \in J'$  so  $n-1 \notin I' \setminus J'$ ). Finally, if  $n-1 \in I$  but  $n-1 \notin J$  then, for the last run  $I_t$  of  $I$ ,  $I_t \cap J$  is not a prefix of  $I_t$ , and thus  $a_{I,J} = 0$  in  $A_n$  as well as  $-b_{I',J'} = 0$  in  $B_{n-1}$  (since  $n-1 \in I' \setminus J'$ ).

We have proved the recursion for  $A_n$ . The entries of  $B_n$  are equal to the corresponding entries of  $A_n$ , except for those in the quarter corresponding to  $(I, J)$  with  $n \in I$  and  $n \notin J$ , which are all zeros (since  $n \in I \setminus J$ ). This proves the recursion for  $B_n$  as well.

□

### 4.3. Determinants.

It turns out that the invertibility of  $A_n$  is the key factor in an answer to Question 3.4.

**Theorem 4.9.**  *$A_n$  and  $B_n$  are invertible for all  $n \geq 0$ . In fact,*

$$\det(A_n) = (n+1) \cdot \prod_{k=1}^n k^{2^{n-1-k}(n+4-k)} \quad (n \geq 2)$$

while  $\det(A_0) = 1$  and  $\det(A_1) = -2$ , and

$$\det(B_n) = \prod_{k=1}^n k^{2^{n-1-k}(n+2-k)} \quad (n \geq 2)$$

while  $\det(B_0) = 1$  and  $\det(B_1) = -1$ .

A proof of Theorem 4.9 will be given in the next section. For comparison,

$$\det(H_n) = 2^{2^{n-1}n} \quad (n \geq 2)$$

with  $\det(H_0) = 1$  and  $\det(H_1) = -2$ .

### 4.4. Eigenvalues.

Having computed the determinants, it is natural to ask for their factors – the eigenvalues of the matrices  $A_n$  and  $B_n$ . The following surprising conjecture is strongly supported by numerical evidence.

**Conjecture 4.10.** *The roots of the characteristic polynomial of  $A_n$  are in  $2 : 1$  correspondence with the compositions of  $n$ : each composition  $\mu = (\mu_1, \dots, \mu_t)$  of  $n$  corresponds to a pair of eigenvalues  $\pm \sqrt{\pi_\mu}$  of  $A_n$ , where*

$$\pi_\mu := \prod_{i=1}^t (\mu_i + 1).$$

Similarly, the roots of the characteristic polynomial of  $B_n$  are in  $2 : 1$  correspondence with the compositions of  $n$ : each composition  $\mu = (\mu_1, \dots, \mu_t)$  of  $n$  corresponds to a pair of eigenvalues  $\pm \sqrt{\pi'_\mu}$  of  $B_n$ , where

$$\pi'_\mu := \prod_{i=1}^{t-1} (\mu_i + 1).$$

It is not difficult to compute the diagonal elements of  $A_n^2$  and  $B_n^2$ . Their values, together with Conjecture 4.10, imply the following conjecture.

**Conjecture 4.11.** *For each of the matrices  $A_n^2$  and  $B_n^2$ , the multiset of eigenvalues (counted by algebraic multiplicity) is equal to the multiset of diagonal elements, which in turn consists of the numbers  $\pi_\mu$  and  $\pi'_\mu$  from Conjecture 4.10 (with doubled multiplicity).*

Conjecture 4.11 is quite surprising, taking into account that  $A_n^2$  and  $B_n^2$  are square matrices which are not symmetric (and can therefore have, conceivably, non-real eigenvalues); and, moreover, are not even diagonalizable (for moderately large values of  $n$ )!

**Example 4.12.**

$$A_3^2 = \begin{pmatrix} 8 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 8 & -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 & -2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 6 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & -2 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 2 & 1 & 1 & 0 & 4 \end{pmatrix}$$

## 5. MÖBIUS INVERSION

In order to prove Theorem 4.9 we need to study some properties of matrices obtained from  $A_n$  and  $B_n$  by Möbius inversion. For other properties see [4].

### 5.1. Auxiliary definitions.

Let us define certain auxiliary families of matrices.

**Definition 5.1.** Define, recursively,

$$Z_n = \begin{pmatrix} Z_{n-1} & Z_{n-1} \\ 0 & Z_{n-1} \end{pmatrix} \quad (n \geq 1)$$

with  $Z_0 = (1)$ , as well as

$$M_n = \begin{pmatrix} M_{n-1} & -M_{n-1} \\ 0 & M_{n-1} \end{pmatrix} \quad (n \geq 1)$$

with  $M_0 = (1)$ .

$Z_n$  is the *zeta matrix* of the poset  $P_n$  with respect to *set inclusion* (not with respect to its linear extension, described in Definition 4.3). Thus  $Z_n = (z_{I,J})_{I,J \in P_n}$  is a square matrix, with entries satisfying

$$z_{I,J} = \begin{cases} 1, & \text{if } I \subseteq J; \\ 0, & \text{otherwise.} \end{cases}$$



$M_n = Z_n^{-1}$  is the corresponding *Möbius matrix*, expressing the Möbius function (see [16]) of the poset  $P_n$ . Thus  $M_n = (m_{I,J})_{I,J \in P_n}$  has entries satisfying

$$m_{I,J} = \begin{cases} (-1)^{|J \setminus I|}, & \text{if } I \subseteq J; \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 5.2.** Denote  $AM_n := A_n M_n$ ,  $BM_n := B_n M_n$  and  $HM_n := H_n M_n$ .

It follows from Definitions 4.1 and 5.1 that

$$(1) \quad AM_n = \begin{pmatrix} AM_{n-1} & 0 \\ AM_{n-1} & -(AM_{n-1} + BM_{n-1}) \end{pmatrix} \quad (n \geq 1)$$

with  $AM_0 = (1)$  and

$$(2) \quad BM_n = \begin{pmatrix} AM_{n-1} & 0 \\ 0 & -BM_{n-1} \end{pmatrix} \quad (n \geq 1)$$

with  $BM_0 = (1)$ , as well as

$$(3) \quad HM_n = \begin{pmatrix} HM_{n-1} & 0 \\ HM_{n-1} & -2HM_{n-1} \end{pmatrix} \quad (n \geq 1)$$

with  $HM_0 = (1)$ .

The block triangular form of  $AM_n$  and block diagonal form of  $BM_n$  facilitate a recursive computation of the determinants of  $A_n$  and  $B_n$ .

## 5.2. A proof of Theorem 4.9.

By recursion (2),

$$\det(BM_n) = \det(AM_{n-1}) \det(-BM_{n-1}) \quad (n \geq 1).$$

Now  $M_n$  is an upper triangular matrix with 1-s on its diagonal, so that

$$\det(M_n) = 1.$$

We conclude that

$$(4) \quad \det(B_n) = \delta_{n-1} \det(A_{n-1}) \det(B_{n-1}) \quad (n \geq 1),$$

where

$$\delta_n = (-1)^{2^n} = \begin{cases} -1, & \text{if } n = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Similarly, for any scalar  $t$ ,

$$AM_n + tBM_n = \begin{pmatrix} (t+1)AM_{n-1} & 0 \\ AM_{n-1} & -AM_{n-1} - (t+1)BM_{n-1} \end{pmatrix} \quad (n \geq 1)$$

and a similar argument yields

$$\det(A_n + tB_n) = \delta_{n-1} \det((t+1)A_{n-1}) \det(A_{n-1} + (t+1)B_{n-1}) \quad (n \geq 1).$$

It follows that

$$\begin{aligned}
\det(A_n) &= \delta_{n-1} \det(A_{n-1}) \det(A_{n-1} + B_{n-1}) \\
&= \delta_{n-1} \det(A_{n-1}) \delta_{n-2} \det(2A_{n-2}) \det(A_{n-2} + 2B_{n-2}) \\
&= \dots \\
&= \left( \prod_{k=1}^n \delta_{n-k} \det(kA_{n-k}) \right) \cdot \det(A_0 + nB_0) = \\
&= -(n+1) \cdot \prod_{k=1}^n k^{2^{n-k}} \cdot \prod_{k=1}^n \det(A_{n-k}) \quad (n \geq 1).
\end{aligned}$$

Since  $A_0 = (1)$  it follows that  $\det(A_n) \neq 0$  for any nonnegative integer  $n$ , and therefore

$$\det(A_n)/\det(A_{n-1}) = \frac{-(n+1)}{-n} \cdot n \cdot \prod_{k=1}^{n-1} k^{2^{n-1-k}} \cdot \det(A_{n-1}) \quad (n \geq 2).$$

The solution to this recursion, with initial value  $\det(A_1) = -2$ , is

$$\det(A_n) = (n+1) \cdot \prod_{k=1}^n k^{2^{n-1-k}(n+4-k)} \quad (n \geq 2).$$

Recursion (4) above, with initial value  $\det(B_1) = -1$ , now yields

$$\det(B_n) = \prod_{k=1}^n k^{2^{n-1-k}(n+2-k)} \quad (n \geq 2).$$

For comparison,

$$\det(H_n) = 2^{2^{n-1}} \det(H_{n-1})^2 \quad (n \geq 2)$$

with initial value  $\det(H_1) = -2$ , so that

$$\det(H_n) = 2^{2^{n-1}n} \quad (n \geq 2).$$

□

**Remark 5.3.** We can also write

$$\det(A_n) = \prod_{k=1}^{n+1} k^{a_{n+1-k}} \quad (n \geq 2),$$

where the sequence  $(a_0, a_1, \dots) = (1, 2, 5, 12, 28, 64, \dots)$  coincides with [17, sequence A045623].

### 5.3. Explicit inverse matrices.

We would like to have explicit expressions for the entries of  $A_n^{-1}$ . This turns out to be difficult to do directly, and we shall compute, as an intermediate step, the entries of  $AM_n^{-1}$ . Note that  $A_n^{-1} = M_n \cdot AM_n^{-1}$ .

**Example 5.4.**

$$A_3^{-1} = \begin{pmatrix} 1/24 & 1/24 & 1/12 & 1/12 & 1/8 & 1/8 & 1/4 & 1/4 \\ 1/8 & -1/24 & 1/12 & -1/12 & 5/24 & -1/8 & 1/12 & -1/4 \\ 5/24 & 5/24 & -1/12 & -1/12 & 1/8 & 1/8 & -1/4 & -1/4 \\ 1/8 & -5/24 & -1/12 & 1/12 & 1/24 & -1/8 & -1/12 & 1/4 \\ 1/8 & 1/8 & 1/4 & 1/4 & -1/8 & -1/8 & -1/4 & -1/4 \\ 5/24 & -1/8 & 1/12 & -1/4 & -5/24 & 1/8 & -1/12 & 1/4 \\ 1/8 & 1/8 & -1/4 & -1/4 & -1/8 & -1/8 & 1/4 & 1/4 \\ 1/24 & -1/8 & -1/12 & 1/4 & -1/24 & 1/8 & 1/12 & -1/4 \end{pmatrix}$$

$$AM_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/6 & -1/3 & -1/6 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \\ 1/4 & -1/4 & 0 & 0 & -1/4 & 1/4 & 0 & 0 \\ 1/6 & 0 & -1/3 & 0 & -1/6 & 0 & 1/3 & 0 \\ 1/24 & -1/8 & -1/12 & 1/4 & -1/24 & 1/8 & 1/12 & -1/4 \end{pmatrix}$$

We shall attempt an inductive computation of  $AM_n^{-1}$ . Recursion formulas (1) and (2) yield corresponding recursions for the inverse matrices:

$$AM_n^{-1} = \begin{pmatrix} AM_{n-1}^{-1} & 0 \\ (AM_{n-1} + BM_{n-1})^{-1} & -(AM_{n-1} + BM_{n-1})^{-1} \end{pmatrix} \quad (n \geq 1)$$

and

$$BM_n^{-1} = \begin{pmatrix} AM_{n-1}^{-1} & 0 \\ 0 & -BM_{n-1}^{-1} \end{pmatrix} \quad (n \geq 1),$$

with  $AM_0^{-1} = BM_0^{-1} = (1)$ ; however, the recursion for  $AM_n^{-1}$  involves the inverse of a new matrix,  $AM_{n-1} + BM_{n-1}$ , which in turn involves the inverse of  $AM_{n-2} + 2BM_{n-2}$ , and so forth. We are thus led to consider a more general situation.

**Definition 5.5.** For any real number  $x$  let

$$M_n(x) := xAM_n + (1-x)BM_n.$$

In particular,  $M_n(0) = BM_n$  and  $M_n(1) = AM_n$ .

**Theorem 5.6.** For each  $n \geq 0$  and  $x > 0$ ,

$$M_n^{-1}(x)_{I,J} \neq 0 \iff J \subseteq I$$

and, for  $J \subseteq I$ ,

$$M_n^{-1}(x)_{I,J} = (-1)^{|J|} \prod_{i \in I} \frac{d_{I,J,x}(i)}{e_{I,J,x}(i)},$$

where  $I_1, \dots, I_t$  are the runs of  $I$  and, for  $i \in I_k$ :

(i) If  $n \notin I_k$  then

$$d_{I,J,x}(i) := \begin{cases} \max(I_k) - i + 1, & \text{if } i \in J; \\ 1, & \text{otherwise} \end{cases}$$

and

$$e_{I,J,x}(i) := \max(I_k) - i + 2.$$

(ii) If  $n \in I_k$  (and thus necessarily  $k = t$ ) then

$$d_{I,J,x}(i) := \begin{cases} (\max(I_k) - i) \cdot x + 1, & \text{if } i \in J; \\ x, & \text{otherwise} \end{cases}$$

and

$$e_{I,J,x}(i) := (\max(I_k) - i + 1) \cdot x + 1.$$

*Proof.* Let  $x > 0$ . By Definition 5.5 and recursion formulas (1) and (2),

$$\begin{aligned} M_n(x) &= \begin{pmatrix} AM_{n-1} & 0 \\ xAM_{n-1} & -(xAM_{n-1} + BM_{n-1}) \end{pmatrix} \\ &= \begin{pmatrix} M_{n-1}(1) & 0 \\ xM_{n-1}(1) & -(1+x)M_{n-1}\left(\frac{x}{1+x}\right) \end{pmatrix} \quad (n \geq 1) \end{aligned}$$

with  $M_0(x) = (1)$ . Invertibility of  $M_{n-1}(x)$  for all  $x > 0$  clearly implies the invertibility of  $M_n(x)$  for all  $x > 0$ . The inverse satisfies

$$(5) \quad M_n^{-1}(x) = \begin{pmatrix} M_{n-1}^{-1}(1) & 0 \\ \frac{x}{1+x}M_{n-1}^{-1}\left(\frac{x}{1+x}\right) & \frac{-1}{1+x}M_{n-1}^{-1}\left(\frac{x}{1+x}\right) \end{pmatrix} \quad (n \geq 1)$$

with  $M_0^{-1}(x) = (1)$ , for all  $x > 0$ .

Recursion (5) shows that, indeed, for  $x > 0$ :  $M_n^{-1}(x)_{I,J} \neq 0 \iff J \subseteq I$ , and that the sign of this entry is  $(-1)^{|J|}$ .

Regarding the absolute value of this entry, assume by induction that the prescribed formula holds for  $M_{n-1}^{-1}(x)$ ,  $\forall x > 0$ .

If  $n \notin I$  then also  $n \notin J$ , and clearly  $M_n^{-1}(x)_{I,J} = M_{n-1}^{-1}(1)_{I,J}$  satisfies the required formula.

If  $n \in I$ , let  $I' := I \setminus \{n\}$ ,  $J' := J \setminus \{n\}$  and  $x' := \frac{x}{1+x}$ . The assumed formula for  $M_{n-1}^{-1}(x')_{I',J'}$  and the claimed formula for  $M_n^{-1}(x)_{I,J}$  have exactly the same factors for all  $i \notin I_t$ , so we need only consider  $i \in I_t$ .

If  $|I_t| = 1$  (i.e.,  $n - 1 \notin I$ ) then there is nothing else in  $M_{n-1}^{-1}(x')_{I',J'}$ , but according to (5) there is an extra factor  $\frac{1}{1+x}$  or  $\frac{x}{1+x}$  in  $M_n^{-1}(x)_{I,J}$  (depending on whether or not  $n \in J$ ), and this is exactly the missing  $d_{I,J,x}(n)/e_{I,J,x}(n)$ .

Finally, assume that  $|I_t| > 1$ . Again, the extra factor  $\frac{1}{1+x}$  or  $\frac{x}{1+x}$  is exactly  $d_{I,J,x}(n)/e_{I,J,x}(n)$ . The other factors in  $M_{n-1}^{-1}(x')_{I',J'}$ , corresponding to  $i \in I'_t$ , are (if  $i \in J$ )

$$\frac{d_{I',J',x'}(i)}{e_{I',J',x'}(i)} = \frac{(n-1-i)x' + 1}{(n-i)x' + 1} = \frac{(n-1-i)x + 1 + x}{(n-i)x + 1 + x} = \frac{d_{I,J,x}(i)}{e_{I,J,x}(i)}$$

or (if  $i \notin J$ )

$$\frac{d_{I',J',x'}(i)}{e_{I',J',x'}(i)} = \frac{x'}{(n-i)x' + 1} = \frac{x}{(n-i)x + 1 + x} = \frac{d_{I,J,x}(i)}{e_{I,J,x}(i)},$$

exactly as claimed for  $M_n^{-1}(x)_{I,J}$ .  $\square$

We are especially interested, of course, in the special case  $x = 1$ .

**Corollary 5.7.** ( $AM_n$  inverse)

For each  $n \geq 0$

$$(AM_n^{-1})_{I,J} \neq 0 \iff J \subseteq I$$

and, for  $J \subseteq I$ ,

$$(AM_n^{-1})_{I,J} = (-1)^{|J|} \prod_{i \in I} \frac{d_{I,J}(i)}{e_{I,J}(i)},$$

where  $I_1, \dots, I_t$  are the runs of  $I$  and, for  $i \in I_k$ :

$$d_{I,J}(i) := \begin{cases} \max(I_k) - i + 1, & \text{if } i \in J; \\ 1, & \text{otherwise} \end{cases}$$

and

$$e_{I,J}(i) := \max(I_k) - i + 2.$$

Equivalently, for  $J \subseteq I$ ,

$$(AM_n^{-1})_{I,J} = (-1)^{|J|} \prod_{k=1}^t \frac{1}{(|I_k| + 1)!} \prod_{i \in I_k \cap J} (\max(I_k) - i + 1).$$

Note that the denominator  $\prod_{k=1}^t (|I_k| + 1)!$  is the cardinality of the parabolic subgroup  $\langle I \rangle$  of  $S_{n+1}$  generated by the simple reflections  $\{s_i : i \in I\}$ .

**Corollary 5.8.**

- (i) Each nonzero entry of  $AM_n^{-1}$  is the inverse of an integer.
- (ii) In each row of  $AM_n^{-1}$ , the sum of absolute values of all the entries is 1.
- (iii) In each row  $I$  of  $AM_n^{-1}$ , the first entry

$$(AM_n^{-1})_{I,\emptyset} = \prod_{k=1}^t \frac{1}{(|I_k| + 1)!}$$

divides all the other nonzero entries and the diagonal entry

$$(AM_n^{-1})_{I,I} = (-1)^{|I|} \prod_{k=1}^t \frac{1}{|I_k| + 1}$$

is divisible by all the other nonzero entries, where a rational number  $r$  is said to divide a rational number  $s$  if the quotient  $s/r$  is an integer.

## 6. FINE SETS

A general setting for character formulas is introduced in this section. It will serve, in the next section, as a framework for the answer to Question 3.4.

Recall from Subsection 2.1 the definition of  $I(\mu)$  for a composition  $\mu$ .

**Definition 6.1.** Let  $\mu = (\mu_1, \dots, \mu_t)$  be a composition of  $n$ . A subset  $J \subseteq [n-1]$  is  $\mu$ -unimodal if each run of  $J \cap I(\mu)$  is a prefix of the corresponding run of  $I(\mu)$ ; in other words, if  $J \cap I(\mu)$  is a disjoint union of intervals of the form  $\left[\sum_{i=1}^{k-1} \mu_i + 1, \sum_{i=1}^{k-1} \mu_i + \ell_k\right]$ , where  $0 \leq \ell_k \leq \mu_k - 1$  for every  $1 \leq k \leq t$ .

**Observation 6.2.** A permutation  $\pi \in S_n$  is  $\mu$ -unimodal according to the definition in Subsection 2.3 if and only if its descent set  $\text{Des}(\pi)$  is  $\mu$ -unimodal according to Definition 6.1.

**Definition 6.3.** Let  $\mathcal{B}$  be a set of combinatorial objects, and let  $\text{Des} : \mathcal{B} \rightarrow P_{n-1}$  be a map which associates with each element  $b \in \mathcal{B}$  a subset  $\text{Des}(b) \subseteq [n-1]$ . Denote by  $\mathcal{B}^\mu$  the set of elements in  $\mathcal{B}$  whose “descent set”  $\text{Des}(b)$  is  $\mu$ -unimodal. Let  $\rho$  be a complex  $S_n$ -representation. Then  $\mathcal{B}$  is called a *fine set* for  $\rho$  if, for each composition  $\mu$  of  $n$ , the character of  $\rho$  at a conjugacy class of cycle type  $\mu$  satisfies

$$(6) \quad \chi_\mu^\rho = \sum_{b \in \mathcal{B}^\mu} (-1)^{|\text{Des}(b) \cap I(\mu)|}.$$

It follows from Theorems 3.1, 3.2 and 3.3 that

**Proposition 6.4.**

- (i) Any Knuth class of RSK shape  $\lambda$  is a fine set for the Specht module  $S^\lambda$ .
- (ii) The set of permutations of a fixed Coxeter length  $k$  in  $S_n$  is a fine set for the  $k$ -th homogeneous component of the coinvariant algebra of  $S_n$ .
- (iii) The set of involutions in  $S_n$  is a fine set for the Gelfand model of  $S_n$ .

Another example of a fine set is given in [7, §9].

The following criterion is useful.

**Proposition 6.5.** Let  $\rho$  be an  $S_n$ -representation, let  $\{C_b : b \in \mathcal{B}\}$  be a basis for the representation space, and let  $\text{Des} : \mathcal{B} \rightarrow P_{n-1}$  be a map. If for every  $1 \leq i \leq n-1$  and  $b, v \in \mathcal{B}$  there are suitable coefficients  $a_i(b, v)$  such that

$$s_i(C_b) = \begin{cases} -C_b, & \text{if } i \in \text{Des}(b); \\ C_b + \sum_{v \in \mathcal{B} \text{ s.t. } i \in \text{Des}(v)} a_i(b, v) C_v, & \text{otherwise} \end{cases}$$

then  $\mathcal{B}$  is a fine set for  $\rho$ .

The proof is a natural extension of the proof of [14, Theorem 1] and is omitted.

Two well known bases which satisfy the assumptions of Proposition 6.5 are the Kazhdan-Lusztig basis for the group algebra [10, (2.3.b), (2.3.d)] and the Schubert polynomial basis for the coinvariant algebra [5, Theorem 3.14(iii)][2]. Since  $S_n$  embeds naturally in classical Weyl groups of rank  $n$ , it follows that Kazhdan-Lusztig cells, as well as subsets of elements of fixed Coxeter length in these groups, are fine sets for the  $S_n$ -action.

## 7. DISTRIBUTION OF DESCENT SETS

### 7.1. Main results.

We are now ready to state our main theorem.

**Theorem 7.1.** *If  $\mathcal{B}$  is a fine set for an  $S_n$ -representation  $\rho$  then the character values of  $\rho$  determine the distribution of descent sets over  $\mathcal{B}$ . In particular, for every  $I \subseteq [n-1]$ , the number of elements in  $\mathcal{B}$  whose descent set contains  $I$  satisfies*

$$|\{b \in \mathcal{B} : I \subseteq \text{Des}(b)\}| = \frac{1}{|\langle I \rangle|} \sum_{J \subseteq I} (-1)^{|J|} \chi^\rho(c_J) \prod_{k=1}^t \prod_{i \in I_k \cap J} (\max(I_k) - i + 1),$$

where  $I_1, \dots, I_t$  are the runs in  $I$ ,  $|\langle I \rangle|$  is the cardinality of the parabolic subgroup of  $S_n$  generated by  $\{s_i : i \in I\}$ , and  $c_I$  is any Coxeter element in this subgroup.

*Proof.* The mapping  $\mu \mapsto I(\mu)$  (see Subsection 2.1) is a bijection between the set of all compositions of  $n$  and the set  $P_{n-1}$  of all subsets of  $[n-1]$ . For a subset  $J = \{j_1, \dots, j_k\} \subseteq [n-1]$  with  $j_1 < j_2 < \dots < j_k$  let  $c_J$  be the product  $s_{j_1} s_{j_2} \dots s_{j_k} \in S_n$ . This is a Coxeter element in the parabolic subgroup generated by  $\{s_i : i \in J\}$ , and its cycle type is (the partition corresponding to) the composition  $\mu$ , where  $J = I(\mu)$ . Let  $x^\rho$  be the vector with entries  $\chi^\rho(c_J)$ , where the subsets  $J \in P_{n-1}$  are ordered anti-lexicographically as in Definition 4.3.

Similarly, let  $v^\mathcal{B} = (v_J^\mathcal{B})_{J \in P_{n-1}}$  be the vector with entries

$$v_J^\mathcal{B} := |\{b \in \mathcal{B} : \text{Des}(b) = J\}| \quad (\forall J \in P_{n-1}).$$

By Definition 6.3 and Lemma 4.8(i),  $\mathcal{B}$  is a fine set for  $\rho$  if and only if

$$(7) \quad x^\rho = A_{n-1} v^\mathcal{B},$$

where  $x^\rho$  and  $v^\mathcal{B}$  are written as column vectors. By Theorem 4.9,  $A_{n-1}$  is an invertible matrix, which proves that  $x^\rho$  uniquely determines  $v^\mathcal{B}$ .

The explicit formula follows from Corollary 5.7, as soon as equation (7) is written in the form

$$Z_{n-1} v^\mathcal{B} = A M_{n-1}^{-1} x^\rho.$$

□

**Example 7.2.** For every  $I \subseteq [n-1]$ , the number of standard Young tableaux of shape  $\lambda$  whose descent set contains  $I$  is equal to

$$\frac{1}{|\langle I \rangle|} \sum_{J \subseteq I} (-1)^{|J|} \chi^\lambda(c_J) \prod_{k=1}^t \prod_{i \in I_k \cap J} (\max(I_k) - i + 1).$$

The Inclusion-Exclusion Principle (namely, multiplication by  $M_{n-1}$ ) gives an equivalent form of the explicit formula.

**Corollary 7.3.** *Let  $B$  be a fine set for an  $S_n$ -representation  $\rho$ . For every  $I \subseteq [n-1]$ , the number of elements in  $B$  with descent set  $D$  satisfies*

$$|\{b \in B : \text{Des}(b) = D\}| = \sum_J \chi^\rho(c_J) \sum_{I: D \cup J \subseteq I} (-1)^{|I \setminus D|} (AM_{n-1}^{-1})_{I,J}$$

where

$$(AM_{n-1}^{-1})_{I,J} = \frac{(-1)^{|J|}}{|\langle I \rangle|} \prod_{k=1}^t \prod_{i \in I_k \cap J} (\max(I_k) - i + 1)$$

and the notation is as in Theorem 7.1.

## 7.2. Permutation statistics versus character theory.

By Theorem 7.1, certain statements in permutation statistics have equivalent statements in character theory. In particular,

**Corollary 7.4.** *Given two symmetric group modules with fine sets, the isomorphism of these modules is equivalent to equi-distribution of the descent set on their fine sets.*

*Proof.* Combining Theorem 7.1 with Definition 6.3.  $\square$

Here is a distinguished example. Recall the major index of a permutation  $\pi$ ,

$$\text{maj}(\pi) := \sum_{i \in \text{Des}(\pi)} i.$$

For a subset  $I \subseteq [n-1]$  denote  $\mathbf{x}^I := \prod_{i \in I} x_i$ . The following is a fundamental theorem in permutation statistics.

**Theorem 7.5.** (Foata-Schützenberger) [8]

$$\sum_{\pi \in S_n} \mathbf{x}^{\text{Des}(\pi)} q^{\ell(\pi)} = \sum_{\pi \in S_n} \mathbf{x}^{\text{Des}(\pi)} q^{\text{maj}(\pi^{-1})}.$$

See also [9].

For  $0 \leq k \leq \binom{n}{2}$  denote by  $R_k$  the  $k$ -th homogeneous component of the coinvariant algebra of the symmetric group  $S_n$ . The following is a classical theorem in invariant theory.



**Theorem 7.6.** (Lusztig-Stanley) [18, Prop. 4.11] *For a partition  $\lambda$  denote by  $m_{k,\lambda}$  the number of standard Young tableaux of shape  $\lambda$  with major index  $k$ . Then*

$$R_k \cong \bigoplus_{\lambda \vdash n} m_{k,\lambda} S^\lambda,$$

where the sum is over all partitions of  $n$  and  $S^\lambda$  denotes the irreducible  $S_n$ -module indexed by  $\lambda$ .

It follows from Corollary 7.4 that

**Corollary 7.7.** *The Foata-Schützenberger Theorem is equivalent to the Lusztig-Stanley Theorem.*

*Proof.* First, notice that the set of permutations  $B_k = \{\pi \in S_n : \text{maj}(\pi^{-1}) = k\}$  is a disjoint union of Knuth classes, where for each partition  $\lambda \vdash n$ , there are exactly  $m_{k,\lambda}$  Knuth classes of RSK shape  $\lambda$  in this disjoint union. Combining this fact with Proposition 6.4(1) implies that  $B_k$  is a fine set for the representation  $\rho_k := \bigoplus_{\lambda \vdash n} m_{k,\lambda} S^\lambda$ .

On the other hand, by Proposition 6.4(2), the set of permutations  $L_k = \{\pi \in S_n : \ell(\pi) = k\}$  is a fine set for  $R_k$ .

Combining these facts with Corollary 7.4,  $\rho_k \cong R_k$  if and only if the distributions of the descent set over  $B_k$  and  $L_k$  are equal. □

**Remark 7.8.** A combinatorial proof of the Lusztig-Stanley Theorem as an application of Foata-Schützenberger’s Theorem appears in [14]. The opposite implication is new.

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